ITEP/TH-23/10

From Brezin-Hikami to Harer-Zagier formulas for Gaussian correlators

A.Morozov and Sh.Shakirov

ITEP, Moscow, Russia

ABSTRACT

Brezin-Hikami contour-integral representation of exponential multidensities in finite N Hermitian matrix model is a remarkable implication of the old Hermitian-Kontsevich duality. It is also a simplified version of Okounkov's formulas for the same multidensities in the cubic Kontsevich model and of Nekrasov calculus for LMNS integrals, a central piece of the modern studies of AGT relations. In this paper we use Brezin-Hikami representation to derive explicit expressions for the Harer-Zagier multidensities (from arXiv:0906.0036): the only known exhaustive generating functions of all-genera Gaussian correlators which are fully calculable and expressed in terms of elementary functions. Using the Brezin-Hikami contour integrals, we rederive the 1-point function of Harer and Zagier and the 2-point arctangent function of arXiv:0906.0036. We also present (without a proof) the explicit expression for the 3-point function in terms of arctangents. Derivation of the 3-point and higher Harer-Zagier functions remains a challenging problem.

Contents

1 Introduction **1**

2 Harer-Zagier formulas: a reminder **4**
2.1 1-point function 5
2.2 2-point function 5
2.3 3-point function 5

3 Harer-Zagier formulas from Brezin-Hikami integrals **6**
3.1 1-point function 6
3.2 2-point function 8

1 Introduction

Gaussian correlators are among the central quantities in mathematical physics. For many decades they remain the subject of constant investigation and innumerable applications, still we are far from a satisfactory and exhaustive description. The problem sounds very simple: to evaluate Gaussian integrals over $N \times N$ Hermitian matrices

$$C_{i_1\ldots i_k}(N) = \ \left\langle \mathrm{tr}\,\phi^{i_1}\ldots \mathrm{tr}\,\phi_{i_k}\right\rangle \ \equiv \frac{\int_{N\times N} \mathrm{tr}\,\phi^{i_1}\ldots \mathrm{tr}\,\phi_{i_k} e^{-\frac{1}{2}\mathrm{tr}\,\phi^2}d\phi}{\int_{N\times N} e^{-\frac{1}{2}\mathrm{tr}\,\phi^2}d\phi} \tag{1}$$

or their *connected* counterparts

$$C_i^{\mathrm{conn}} = \ \left\langle\left\langle \mathrm{tr}\,\phi^i\right\rangle\right\rangle \ = C_i \tag{2}$$

$$C_{ij}^{\mathrm{conn}} = \ \left\langle\left\langle \mathrm{tr}\,\phi^i \mathrm{tr}\,\phi^j\right\rangle\right\rangle \ = C_{ij} - C_iC_j \tag{3}$$

$$C_{ijk}^{\mathrm{conn}} = \ \left\langle\left\langle \mathrm{tr}\,\phi^i \mathrm{tr}\,\phi^j \mathrm{tr}\,\phi^k\right\rangle\right\rangle \ = C_{ijk} - C_{ij}C_k - C_{ik}C_j - C_{jk}C_i + 2C_iC_jC_k \tag{4}$$

$$\ldots\ldots\ldots$$

and it is an easy exercise to find any particular quantity of this kind, for particular low values of $i_1, \ldots, i_k$

$$\begin{aligned} C_0(N) &= N \\ C_2(N) &= N^2 \\ C_4(N) &= 2N^3 + N, \\ C_6(N) &= 5N^4 + 10N^2, \\ C_{22}(N) &= N^4 + 2N^2, \\ C_{222}(N) &= N^6 + 6N^4 + 8N^2, \end{aligned} \tag{5}$$

or, say, for particular low values of N:

$$C_{k_1,\ldots,k_m}(1) = (k_1 + \ldots + k_m - 1)!! \quad \text{if } k_1 + \ldots + k_m \text{ is even, and 0 otherwise} \tag{6}$$

The fact that each connected correlator is a non-trivial polynomial in N is encoded in the idea of a *genus expansion*: different powers of N come from the fat-graph Feynman diagrams of different topology. The real problem is to find a closed expression for *generic* C and $C^{\rm conn}$ or, at least, for contributions of particular genera.

As usual, straightforward approach is to construct a generating function and evaluate it exactly. Also as usual, resolvability of this problem depends on the *clever* choice of the generating function. The simplest choice is to consider *resolvents*

$$\rho(z_1, \ldots, z_k) = \frac{\nabla(z_1)\ldots\nabla(z_k)\mathcal{Z}}{\mathcal{Z}} \equiv \sum_{i_1,\ldots,i_k=0}^{\infty} \frac{C_{i_1\ldots i_k}(N)}{z_1^{i_1+1}\ldots z_k^{i_k+1}} = \left\langle \operatorname{tr}\frac{1}{z_1-\phi}\ldots\operatorname{tr}\frac{1}{z_k-\phi}\right\rangle \tag{7}$$

or

$$\rho^{\rm conn}(z_1, \ldots, z_k) = \nabla(z_1)\ldots\nabla(z_k)\log\mathcal{Z} \equiv \sum_{i_1,\ldots,i_k=0}^{\infty} \frac{C^{\rm conn}_{i_1\ldots i_k}(N)}{z_1^{i_1+1}\ldots z_k^{i_k+1}} = \left\langle\left\langle \operatorname{tr}\frac{1}{z_1-\phi}\ldots\operatorname{tr}\frac{1}{z_k-\phi}\right\rangle\right\rangle \tag{8}$$

These generating functions are very well adjusted to the use of Ward indentities for Gaussian matrix integrals, also known under the names of loop equations or Virasoro constraints, see [1] for a comprehensive review. The problem is, however, that only contributions of particular genera can be calculated in a closed form this way [2]:

$$\begin{aligned} &\rho_0^{\rm conn}(z) = \frac{z - y_N(z)}{2}, \quad \rho_1^{\rm conn}(z) = \frac{N}{y_N^5(z)}, \quad \rho_2^{\rm conn}(z) = \frac{21N(z^2+N)}{y_N^{11}(z)}, \\ &\rho_0^{\rm conn}(z_1, z_2) = \frac{1}{2(z_1-z_2)^2}\left(\frac{z_1z_2 - 4N}{y_N(z_1)y_N(z_2)} - 1\right), \\ &\rho_0^{\rm conn}(z_1, z_2, z_3) = \frac{2N(z_1z_2 + z_1z_3 + z_2z_3 + 4N)}{y_N(z_1)^3 y_N(z_2)^3 y_N(z_3)^3} \end{aligned} \tag{9}$$

where $y_N^2(z) = z^2 - 4N$ is what is called a spectral curve of the (in this case, Gaussian) matrix model. Given a spectral curve, Virasoro constraints can be re-interpreted as "topological recursion" rules [**?**, **?**, **?**], allowing to build $\rho_g(z_1, \ldots, z_n)$ one after another, recursively in the numbers of handles and punctures g and n.

As one calculates more and more resolvents of particular genera, it starts getting clear that formulas become increasingly complicated: it is rather hard to imagine that generic expression for ρ or $\rho^{\rm conn}$ can be obtained in any closed form. It turns out that a minor modification of the generating function makes it calculable! Moreover, these *Harer-Zagier functions* turn to be *elementary* functions. In [3] actually only a 1-point function was introduced

$$\phi(z|N) \equiv \sum_{i=0}^{\infty} \frac{z^{2i}}{(2i-1)!!}\left\langle \operatorname{tr}\phi^{2i}\right\rangle = \frac{1}{2z^2}\left(\left(\frac{1+z^2}{1-z^2}\right)^N - 1\right) \tag{10}$$

and the difference from $\rho(z)$ is the peculiar bifactorial in the denominator. As shown in [4], this result is by no means limited to a 1-point function: all Harer-Zagier functions $\phi(z_1, \ldots, z_n)$ look comprehensible! To make expressions really elementary one should also get rid of N, by making additional generating function [2, 4]:

$$\hat\phi(z|\lambda) \equiv \sum_{N=0}^{\infty} \phi(z|N)\lambda^N = \frac{1}{2z^2}\left(\frac{1}{1-\lambda\frac{1+z^2}{1-z^2}} - \frac{1}{1-\lambda}\right) = \frac{\lambda}{(1-\lambda)}\frac{1}{1-\lambda-(1+\lambda)z^2} \tag{11}$$

Then, for example [4],

$$\hat\phi_{\rm odd}(z_1,z_2|\lambda)\equiv\sum_{N=0}^\infty\lambda^N\sum_{i_1,i_2=0}^\infty\frac{z^{2i_1+1}}{(2i_1+1)!!}\frac{z^{2i_2+1}}{(2i_2+1)!!}\,C_{2i_1+1,2i_2+1}(N)=\frac{\lambda}{(1-\lambda)^{3/2}}\frac{\arctan\frac{z_1z_2\sqrt{1-\lambda}}{\sqrt{1-\lambda+(1+\lambda)(z_1^2+z_2^2)}}}{\sqrt{1-\lambda+(1+\lambda)(z_1^2+z_2^2)}} \quad (12)$$

and so on, see s.2 for more examples. The purpose of this paper is to shed new light to calculability of such Harer-Zagier functions and make a step towards derivation of closed and generic expressions for *all* of them. The idea is to use a still another, third, type of the generating function,

$$e(s_1,\ldots,s_k)=\ \Big\langle{\rm tr}\,e^{s_1\phi}\ldots{\rm tr}\,e^{s_k\phi}\Big\rangle \quad (13)$$

and

$$e^{\rm conn}(s_1,\ldots,s_k)=\ \Big\langle\Big\langle{\rm tr}\,e^{s_1\phi}\ldots{\rm tr}\,e^{s_k\phi}\Big\rangle\Big\rangle \quad (14)$$

They turn out to be closely related to (Generalized) *Kontsevich matrix model* (GKM) [5, 1], which in Gaussian case possesses a remarkable *duality* [6] with the Gaussian matrix model, considered in the present paper. This duality led E.Breain and S.Hikami to a wonderful representation of $e(s|N)$ in the contour integral form [7]:

$$e(s_1,\ldots,s_k|N)=\prod_{i=1}^k\frac{e^{\frac{s_i^2}{2}}}{s_i}\oint du_ie^{u_is_i}\left(1+\frac{s_i}{u_i}\right)^N\det_{1\le i,j\le k}\frac{1}{u_i-(u_j+s_j)} \quad (15)$$

The last determinantal factor here can be also rewritten as

$$\det_{1\le i,j\le k}\frac{1}{u_i-(u_j+s_j)}=\prod_{i>j}\frac{(u_i-u_j)(u_i-u_j+s_i-s_j)}{(u_i-u_j+s_i)(u_i-u_j-s_j)} \quad (16)$$

what makes the whole expression similar to LMNS integral [8], which recently attracted a lot of new attention in the context of AGT-relations [9]: as already emphasized in [10] duality of [6, 7] should play a prominent role in the understanding of AGT relations, at least on the lines of the Dotsenko-Fateev-matrix-model approach [11]. Remarkably, eq.(15) possesses a direct generalization at least to *cubic* Kontsevich model [12], and questions now arise about further generalizations to arbitrary GKM and about the possible AGT-like applications of these generalizations.

In this paper, however, we do not proceed in these very appealing directions. Instead, we concentrate on the application of (15) to evaluation of Harer-Zagier functions, which – in variance with $\rho(z)$ and $e(s)$ – are elementary fully calculable functions. Application of (15) for this purpose will be absolutely straightforward. First, conversion from N to λ substitutes the factor

$$\prod_{i=1}^k\left(1+\frac{s_i}{u_i}\right)^N$$

by

$$\left(1-\lambda\prod_{i=1}^k\left(1+\frac{s_i}{u_i}\right)\right)^{-1}$$

Second, contour integrals just pick up contributions from poles of the simple rational function of u_i – in a very similar way to Nekrasov's celebrated calculation [13]. Third, the Harer-Zagier bifactorials are made from factorials in $e^{s\phi}$ by shifting $s_i\to s_it_i$ and introducing additional semi-integer-Γ-function factors with the help of t-integrations $\int_0^\infty e^{-t_i^2/2}dt_i$.

In this way we reproduce the 1-point and 2-point Harer-Zagier functions from the Brezin-Hikami integrals. Generalization to 3-point and higher functions should be straightforward, but a clever method needs to be invented to convert the resulting multiple integrals into elementary functions (which, as we conjectured in [4], are always elementary functions of arctangent type). This conversion is trivial in the 1-point case; in the 2-point case for the purpose of such conversion we use series representations, which generalize the elementary formula

$$\frac{\arctan\frac{c}{\sqrt{ab-c^2}}}{\sqrt{ab-c^2}}=\sum_k\frac{4^k(k!)^2}{(2k+1)!}\frac{c^{2k+1}}{(ab)^{k+1}}=\int_0^\infty\int_0^\infty e^{-at_1^2-bt_2^2}\sin(2ct_1t_2)dt_1dt_2 \quad (17)$$

see s.3.2. It is not yet clear if this series approach is really promising for 3-point and higher cases. Probably, some new ideas are required to reveal the general structure of k-point Harer-Zagier elementary functions for $k\ge3$.

2 Harer-Zagier formulas: a reminder

Harer-Zagier functions (multidensities) in the finite N Gaussian model are defined as

$$\phi(s_1,\ldots,s_k|N) = \sum_{i_1,\ldots,i_k=0}^{\infty} C^{\rm conn}_{i_1\ldots i_k}(N)\frac{s_1^{i_1}}{n(i_1)}\ldots\frac{s_k^{i_k}}{n(i_k)} \tag{18}$$

(we will consider only connected Harer-Zagier functions here) where the bifactorial normalisation is

$$n(i) = \left\{ \begin{array}{c} (i-1)!!, \quad i = \text{ even} \\ i!!, \quad i = \text{ odd} \end{array}\right. \tag{19}$$

As emphasized in the previous section, it turns out to be convenient to take a generating function with respect to the size of a matrix N as well, converting it into a continuous parameter λ:

$$\hat\phi(s_1,\ldots,s_k|\lambda) = \sum_{N=0}^{\infty}\lambda^N\phi(s_1,\ldots,s_k|N) \tag{20}$$

Technically, this conversion is done with the following well-known finite and infinite summation formulas: since

$$\frac{1}{(1-\lambda)^{k+1}} = \sum_N \frac{(N+k)!}{N!k!}\lambda^N \tag{21}$$

we have:

$$\begin{array}{r} \displaystyle\sum_N \lambda^N = \frac{1}{1-\lambda}, \\ \displaystyle\sum_N N\lambda^N = \frac{\lambda}{(1-\lambda)^2}, \\ \displaystyle\sum_N N^2\lambda^N = \frac{\lambda(1+\lambda)}{(1-\lambda)^3}, \\ \displaystyle\sum_N N^3\lambda^N = \frac{\lambda(1+4\lambda+\lambda^2)}{(1-\lambda)^4}, \\ \displaystyle\sum_N N^4\lambda^N = \frac{\lambda(1+11\lambda+11\lambda^2+\lambda^3)}{(1-\lambda)^5}, \\ \ldots \end{array} \tag{22}$$

Similarly, from an obvious formula

$$\sum_{j=0}^{n} j(j-1)(j-2)\ldots(j-m) = \frac{(n+1)n(n-1)(n-2)\ldots(n-m)}{m+2} \tag{23}$$

(l.h.s. is a polynomial of degree $m+1$ in n, which obviously vanishes at $n = 0,1,2,\ldots,m$) it follows that

$$\begin{array}{r} \displaystyle\sum_{j=0}^{n} j = \frac{n(n+1)}{2}, \\ \displaystyle\sum_{j=0}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}, \\ \displaystyle\sum_{j=0}^{n} j^3 = \frac{n^2(n+1)^2}{4}, \\ \displaystyle\sum_{j=0}^{n} j^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}\ldots \end{array} \tag{24}$$

2.1 1-point function

The 1-point Harer-Zagier density has a form [2, 3, 4]

$$\hat{\phi}(s|\lambda) = \frac{\lambda}{(1-\lambda)} \frac{1}{1-\lambda-(1+\lambda)s^2} \tag{25}$$

This simple and beautiful formula admits generalizations to 2-point and higher functions, see [4] and below.

2.2 2-point function

The 2-point Harer-Zagier density has a form

$$\hat{\phi}(s_1,s_2|\lambda) = \hat{\phi}_{even}(s_1,s_2|\lambda) + \hat{\phi}_{odd}(s_1,s_2|\lambda) \tag{26}$$

where a convenient separation of the even and odd parts is introduced, and

$$\hat{\phi}_{odd}(s_1,s_2|\lambda) = \frac{\lambda}{(1-\lambda)^{3/2}} \frac{\arctan \frac{z_1 z_2 \sqrt{1-\lambda}}{\sqrt{1-\lambda+(1+\lambda)(z_1^2+z_2^2)}}}{\sqrt{1-\lambda+(1+\lambda)(z_1^2+z_2^2)}} \tag{27}$$

$$\hat{\phi}_{even}(s_1,s_2|\lambda) = \frac{s_1 s_2}{s_1^2-s_2^2}\left(s_1\frac{\partial}{\partial s_1} - s_2\frac{\partial}{\partial s_2}\right)\hat{\phi}_{odd}(s_1,s_2|\lambda) \tag{28}$$

These formulas were derived in [4] with the help of integrable Toda equations. In the present paper we rederive them, using a different method (Brezin-Hikami contour integrals) see below.

2.3 3-point function

The 3-point Harer-Zagier density has a form

$$\hat{\phi}(s_1,s_2,s_3|\lambda) = \hat{\phi}_{even}(s_1,s_2,s_3|\lambda) + \hat{\phi}_{mix}(s_1,s_2,s_3|\lambda) \tag{29}$$

where $\hat{\phi}_{even}(s_1,s_2,s_3|\lambda)$ is a totally even (even w.r.t each of its arguments) part of the function. We present the following expression for this totally even part (without a proof, at conjectural level):

$$\begin{aligned} f\big(\tau \mid s_1,s_2,s_3\big) \;=\; & \left(\frac{1-s_1}{(1+\tau^2 s_1(s_2+s_3))^2} + \frac{1+s_2+s_3-s_1}{1+\tau^2 s_1(s_2+s_3)}\right) \frac{\arctan\left(\sqrt{\frac{s_2 s_3(1-s_1)\tau^2}{1+\tau^2 s_1(s_2+s_3)}}\right)}{\sqrt{\frac{s_2 s_3(1-s_1)\tau^2}{1+\tau^2 s_1(s_2+s_3)}}} + \\ & + \left(\frac{1-s_2}{(1+\tau^2 s_2(s_1+s_3))^2} + \frac{1+s_1+s_3-s_2}{1+\tau^2 s_2(s_1+s_3)}\right) \frac{\arctan\left(\sqrt{\frac{s_1 s_3(1-s_2)\tau^2}{1+\tau^2 s_2(s_1+s_3)}}\right)}{\sqrt{\frac{s_1 s_3(1-s_2)\tau^2}{1+\tau^2 s_2(s_1+s_3)}}} + \\ & + \left(\frac{1-s_3}{(1+\tau^2 s_3(s_1+s_2))^2} + \frac{1+s_1+s_2-s_3}{1+\tau^2 s_3(s_1+s_2)}\right) \frac{\arctan\left(\sqrt{\frac{s_1 s_2(1-s_3)\tau^2}{1+\tau^2 s_3(s_1+s_2)}}\right)}{\sqrt{\frac{s_1 s_2(1-s_3)\tau^2}{1+\tau^2 s_3(s_1+s_2)}}} - \end{aligned}$$

$$-\frac{s_1(s_2+s_3)}{s_2s_3(1+\tau^2 s_1(s_2+s_3))}-\frac{s_2(s_1+s_3)}{s_1s_3(1+\tau^2 s_2(s_1+s_3))}-\frac{s_3(s_1+s_2)}{s_1s_2(1+\tau^2 s_3(s_1+s_2))}+$$

$$+\frac{(s_1+s_2)(s_1+s_3)(s_2+s_3)}{s_1s_2s_3(1+\tau^2(s_1s_2+s_1s_3+s_2s_3-s_1s_2s_3))} \tag{30}$$

with a notation

$$f\big(\tau \mid s_1,s_2,s_3\big)=\frac{4}{1-\tau^2}\frac{(1-s_1-s_2-s_3)^3}{s_1s_2s_3}\ \hat\phi_{even}\left(Ts_1,Ts_2,Ts_3\Big|\frac{1-T}{1+T}\right), \quad T=\tau\sqrt{1-s_1-s_2-s_3} \tag{31}$$

This formula was not presented in [4], but it can also be deduced by the method of computer experiment, exploited in that paper, which becomes today a powerful approach to derivation of important results and conjectures in mathematical physics. It would be interesting to prove (derive) this formula, by using either Toda equations or Brezin-Hikami integrals. It is even more interesting to generalize to further generalize it to the 4-point and higher functions, and check the appealing conjecture that *all* the Harer-Zagier functions are expressed through actangents, as linear combinations, reminiscent of the Okounkov's eq. (**??**), which we remind in the Appendix.

3 Harer-Zagier formulas from Brezin-Hikami integrals

3.1 1-point function

In the case of a 1-point function (15) is especially easy to derive [7], however already in this case the calculation is actually done in Gaussian Kontsevich model, not in Hermitian model itself – thus emphasizing the role of Hermitian-Kontsevich duality [6].

$$\frac{1}{V_N}\int_{N\times N}\operatorname{tr} e^{s\phi}e^{-\operatorname{tr}\phi^2/2}e^{\operatorname{tr}A\phi}d\phi=\sum_k\int e^{s\phi_k}\Delta^2(\phi)\frac{\det e^{\phi_iA_j}}{\Delta(\phi)\Delta(A)}\prod_i e^{-\phi_i^2/2}d\phi_i=$$

$$=\frac{1}{\Delta(A)}\sum_k\int\Delta(\phi)e^{s\phi_k}\sum_P(-)^P\prod_i e^{-\phi_i^2/2+\phi_iA_{P(i)}}d\phi_i=\frac{1}{V_N}\sum_k\frac{\Delta(\tilde A_{(k)})}{\Delta(A)}e^{\operatorname{tr}\tilde A_{(k)}^2} \tag{32}$$

where $V_N=\mathrm{Vol}_{SU(N)}$ is the volume of unitary group and $\tilde A$ is the matrix with eigenvalues

$$\tilde A_i^{(k)}=A_i+s\delta_{ki} \tag{33}$$

This is an elementary essentially rational expression, however, it looks singular in the limit of $A\to 0$. The limit can be easily taken if the formula is rewritten in a somewhat more complicated form of a contour integral [7]:

$$\sum_k\prod_{i<j}\frac{a_i-a_j+s(\delta_{ik}-\delta_{jk})}{a_i-a_j}\prod_i e^{(a_i+s\delta_{ik})^2/2}=e^{s^2/2}\prod_i e^{a_i^2/2}\sum_k e^{sa_k}\prod_{i\neq k}\frac{a_i-a_k-s}{a_i-a_k}=$$

$$=e^{s^2/2}\prod_i e^{a_i^2/2}\oint e^{us}du\prod_i\frac{u-a_i+s}{u-a_i} \tag{34}$$

For $A=0$ this turns into

$$\frac{1}{s}e^{s^2/2}\oint e^{us}\left(1+\frac{s}{u}\right)^N du \tag{35}$$

Following [4] one can now pass to a generating function in N: $f(N)\to\hat f(\lambda)=\sum_N\lambda^N f(N)$. This could be done in different ways, for example, one could write $\sum_N\frac{\lambda^N}{N!}f(N)$ instead. Our choice, however, makes calculation of contour integrals most convenient and elementary. Indeed, instead of (35) we obtain:

$$\hat e(s|\lambda)=\frac{e^{s^2/2}}{s}\oint\frac{e^{us}du}{1-\lambda\left(1+\frac{s}{u}\right)}=\boxed{\frac{\lambda}{(1-\lambda)^2}e^{\frac{1+\lambda}{1-\lambda}\frac{s^2}{2}}} \tag{36}$$

where contour integral picks up a contribution from a single pole at $u=\frac{\lambda s}{1-\lambda}$.

Harer-Zagier function differs by the substitution

$$e(s) = \sum_k \frac{s^{2k}}{(2k)!}\left\langle \mathrm{tr}\,\phi^{2k}\right\rangle \;\longrightarrow\; \phi(s) = \sum_k \frac{s^{2k}}{(2k-1)!!}\left\langle \mathrm{tr}\,\phi^{2k}\right\rangle \tag{37}$$

Since

$$\frac{1}{(2k-1)!!} = \frac{2^k k!}{(2k)!} = \frac{1}{(2k)!}\int_0^\infty t^{2k+1}e^{-t^2/2}dt \tag{38}$$

we have

$$\hat{\phi}(s|\lambda) = \int_0^\infty \hat{e}(st|\lambda)e^{-t^2/2}tdt = \frac{\lambda}{(1-\lambda)^2}\int_0^\infty e^{-\frac{1}{2}t^2\left(1-\frac{1+\lambda}{1-\lambda}s^2\right)}tdt = \boxed{\frac{\lambda}{(1-\lambda)\Big(1-\lambda-(1+\lambda)s^2\Big)}} \tag{39}$$

what successfully reproduces (11).

Note that in this case both exponential and Harer-Zagier generating functions are elementary. In fact from (36) one can also obtain an explicit formula for the 1-point Gaussian resolvent:

$$\hat{\rho}(z|\lambda) \equiv \left\langle \mathrm{tr}\,\frac{1}{z-\phi}\right\rangle = \int_0^\infty \hat{e}(s|\lambda)e^{-sz}ds = \frac{\lambda}{(1-\lambda)^2}\int_0^\infty e^{\frac{1+\lambda}{1-\lambda}\frac{s^2}{2}-sz}ds = \frac{i\lambda}{(1-\lambda)\sqrt{1-\lambda^2}}\ \mathrm{erf}\left(iz\sqrt{\frac{1-\lambda}{1+\lambda}}\right) \tag{40}$$

We need here an expansion of error-function

$$\mathrm{erf}(z) \equiv \int_0^\infty e^{-\frac{x^2}{2}-xz}dx = e^{\frac{z^2}{2}}\int_z^\infty e^{-\frac{x^2}{2}}dx \overset{x^2=z^2+t}{=} \frac{1}{z}\int_0^\infty \frac{e^{-t}dt}{\sqrt{1+\frac{2t}{z^2}}} \tag{41}$$

at *large* values of its argument, where it becomes a series in inverse powers of z. Adequate for working out this asymptotics is the last representation, which immediately implies:

$$\boxed{\hat{\rho}(z|\lambda) = \sum_{k=0}\frac{\lambda(1+\lambda)^k}{(1-\lambda)^{k+2}}\frac{(2k-1)!!}{z^{2k+1}}} \tag{42}$$

Using (22) one can easily check the consistency of this formula with the well known first terms of the ρ-expansion:

$$\rho(z) = \frac{<\mathrm{tr}\,I>}{z} + \frac{<\mathrm{tr}\,\phi^2>}{z^3} + \frac{<\mathrm{tr}\,\phi^4>}{z^5} + \ldots = \frac{N}{z} + \frac{N^2}{z^3} + \frac{2N^3+N}{z^5} + \ldots \tag{43}$$

It is also clear from (42) that the resolvent, including the contributions of all genera is represented by asymptotic, nowhere convergent series, and is not expressed through elementary functions. Expressible are particular terms $\rho^{(p|1)}(z)$ of genus expansion [2],

$$\rho(z) = \frac{z-y(z)}{2} + \frac{N}{y^5(z)} + \frac{21N(z^2+N)}{y^{11}(z)} + \frac{11N(135z^4+558Nz^2+158N^2)}{y^{17}(z)} + \ldots = \frac{N}{z} + \frac{N^2}{z^3} + \frac{2N^3+N}{z^5} + \ldots \tag{44}$$

with $y^2(z) = z^2 - 4N$. Particular terms of the genus expansion (44) can be directly deduced from the general

formula (42). For this it should be re-expanded in monomials $(1-\lambda)^{-s}$:

$$\hat\rho(z|\lambda) = \sum_{k=0} \frac{\lambda(1+\lambda)^k}{(1-\lambda)^{k+2}}\frac{(2k-1)!!}{z^{2k+1}} = \sum_k\sum_{s=0}^\infty \frac{(2k-1)!!}{z^{2k+1}}\frac{2^k}{(1-\lambda)^{s+1}}\oint_0 \frac{(1-\xi)(1-\xi/2)^k}{\xi^{k+2}}\xi^s d\xi =$$

$$\stackrel{(21)}{=} \sum_k \frac{2^k(2k-1)!!}{z^{2k+1}}\sum_s\sum_N \frac{(N+s)!}{N!s!}\lambda^N\oint_0 \frac{(1-\xi)(1-\xi/2)^k}{\xi^{k+2}}\xi^s d\xi \stackrel{(21)}{=} \sum_k \frac{2^k(2k-1)!!}{z^{2k+1}}\sum_N \lambda^N \oint_0 \frac{(1-\xi/2)^k}{(1-\xi)^N\xi^{k+2}}d\xi =$$

$$= \sum_N \lambda^N \sum_k \frac{2^k(2k-1)!!}{z^{2k+1}(k+1)!}\,\partial_\xi^{k+1}\frac{(1-\xi/2)^k}{(1-\xi)^N}\Big|_{\xi=0} =$$

$$= \sum_N\lambda^N\sum_k\frac{2^k(2k-1)!!}{z^{2k+1}}\left(\frac{N^{k+1}}{(k+1)!}+\frac{N^{k-1}}{12(k-2)!}+\frac{N^{k-3}(5k-2)}{1440(k-4)!}+\frac{N^{k-5}(35k^2-77k+12)}{128\cdot 81\cdot 35\cdot(k-6)!}+\ldots\right) =$$

$$= \sum_N\lambda^N\sum_k\frac{2^k(2k-1)!!}{z^{2k+1}}\left\{\frac{N^{k+1}}{(k+1)!}+\frac{N^{k-1}}{12(k-2)!}+\left(\frac{N^{k-3}}{288(k-5)!}+\frac{N^{k-3}}{80(k-4)!}\right)+\right.$$

$$\left.+\left(\frac{N^{k-5}}{128\cdot 81\cdot(k-8)!}+\frac{N^{k-5}}{960\cdot(k-7)!}+\frac{N^{k-5}}{64\cdot 7\cdot(k-6)!}\right)+\ldots\right\} =$$

$$= \sum_N\lambda^N\left(\frac{z-y(z)}{2}+\frac{N}{y^5(z)}+\frac{21N(z^2+N)}{y^{11}(z)}+\frac{11N(135z^4+558Nz^2+158N^2)}{y^{17}(z)}+\ldots\right) \quad (45)$$

In this manipulation eq.(21) is used twice: first in order to convert $(1-\lambda)^{-s-1}$ into a power series in λ and second to find a sum over s, which is equal to $(1-\xi)^{-N-1}$. The integration variable ξ is similar to $1-\lambda$.

Starting from 2-point functions, only Harer-Zagier functions are expressible in terms of elementary functions. Like more general Okounkov's formulas for cubic Kontsevich model, expressions for exponential correlators remain well-defined (convergent), but can be reduced only to integrals of elementary functions. Transition from resolvents to convergent exponential correlators is nothing but the application of the standard Pade summation method, while transition to Harer-Zagier correlators is a fancy modification of Pade method, leading – at least in the case of Gaussian correlators – not only to *convergent*, but to *elementary* expressions. In fact transition between the Harer-Zagier and exponential functions is also an ordinary Pade transform, only with a twice smaller parameter:

$$\hat\rho(z) = \frac{A}{z}\sum_k(2k-1)!!B^k \quad\longrightarrow\quad \hat\phi(s) = A\sum_k B^k = \frac{A}{1-B} \quad\longrightarrow\quad \hat e(s) = A\sum_k\frac{B^k}{2^kk!} = Ae^{B/2} \quad (46)$$

with $A=\frac{\lambda}{(1-\lambda^2)}$, $B=\frac{1+\lambda}{(1-\lambda)}s^2$ and $z=s^{-1}$.

3.2 2-point function

Expression (47) remains almost the same for arbitrary exponential correlators, only for a correlator of n exponentials the label k at the r.h.s. is a vector of the size n and

$$e(s_1,\ldots,s_n|N) = \frac{1}{V_N}\int_{N\times N}\prod_{\alpha=1}^n \mathrm{tr}\, e^{s_\alpha\phi}e^{-\mathrm{tr}\,\phi^2/2}e^{\mathrm{tr}\,A\phi}d\phi = \frac{1}{V_N}\sum_{\vec k}\frac{\Delta(\tilde A_{(\vec k)})}{\Delta(A)}e^{\mathrm{tr}\,\tilde A^2_{(\vec k)}} \quad (47)$$

and

$$\tilde A_i^{(\vec k)} = A_i + \sum_{\alpha=1}^n s_\alpha\delta_{ik_\alpha} \quad (48)$$

Contour-integral representation, however, becomes a little more involved [7]:

$$e(s_1,\ldots,s_n|N) = \prod_{\alpha=1}^n\frac{e^{s_\alpha^2/2}}{s_\alpha}\oint du_\alpha e^{s_\alpha u_\alpha}\left(1+\frac{s_\alpha}{u_\alpha}\right)^N\prod_{\alpha<\beta}\frac{(u_\alpha-u_\beta)(u_\alpha-u_\beta+s_\alpha-s_\beta)}{(u_\alpha-u_\beta+s_\alpha)(u_\alpha-u_\beta-s_\beta)}, \quad (49)$$

so that

$$\hat e(s_1,\ldots,s_n|\lambda) = \prod_{\alpha=1}^n\frac{e^{s_\alpha^2/2}}{s_\alpha}\oint du_\alpha e^{s_\alpha u_\alpha}\frac{\prod_\alpha u_\alpha}{\prod_\alpha u_\alpha - \lambda\prod_\alpha(u_\alpha-s_\alpha)}\prod_{\alpha<\beta}\frac{(u_\alpha-u_\beta)(u_\alpha-u_\beta+s_\alpha-s_\beta)}{(u_\alpha-u_\beta+s_\alpha)(u_\alpha-u_\beta-s_\beta)} \quad (50)$$

In the case of a two-point function, $n = 2$ the integral is two-fold:

$$\hat{e}(s_1, s_2|\lambda) = \frac{e^{(s_1^2+s_2^2)/2}}{s_1 s_2} \oint\oint \frac{e^{s_1 u + s_2 v} uv du dv}{\Big((1-\lambda)uv - \lambda u s_2 - \lambda v s_1 - \lambda s_1 s_2\Big)} \frac{(u-v)(u-v+s_1-s_2)}{(u-v+s_1)(u-v-s_2)} \tag{51}$$

and the main difficulty comes from quadratic factor in the denominator. It is actually linear both in u and v, thus the first contour integral, say, over v is simple: it picks up contributions from the three poles $v = u + s_1$, $v = u - s_1$ and $v = \frac{\lambda s_2 (u+s_1)}{(1-\lambda)u - \lambda s_1}$. However, the last of the three gives rise to a new pole in u, and actually this is the only pole which contributes to the second integral over u: contributions of other poles in u cancel to zero. After all these intermediate calculations, the resulting expression for the exponential 2-point function is

$$\hat{e}(s_1, s_2|\lambda) = \frac{s_1 \lambda^2}{(1-\lambda)^4} \oint_{u=0} \frac{du}{u^2} \exp\left(\frac{(1+\lambda)(s_1^2+s_2^2)}{2(1-\lambda)} + \frac{\lambda s_1 s_2^2}{u(1-\lambda)^2} + s_1 u \right) \times$$

$$\times \frac{\left(1 - \frac{u(1-\lambda)(s_1-s_2)}{s_1 s_2} - \frac{u^2(1-\lambda)^2}{\lambda s_1 s_2}\right)\left(1 - \frac{u(1-\lambda)(s_1-s_2)}{\lambda s_1 s_2} - \frac{u^2(1-\lambda)^2}{\lambda s_1 s_2}\right)}{\left(1 - \frac{u(1-\lambda)}{s_2}\right)\left(1 - \frac{u(1-\lambda)}{\lambda s_2}\right)}$$

and it is not so easy to evaluate, because u^{-1} is present in the exponent. The problem is slightly simplified by taking the odd part of the function, $\hat{e}_{odd}(s_1, s_2|\lambda)$, which takes a somewhat more concise form

$$\hat{e}_{odd}(s_1, s_2|\lambda) = \frac{1}{4}\Big(\hat{e}(s_1, s_2|\lambda) - \hat{e}(s_1, -s_2|\lambda) - \hat{e}(-s_1, s_2|\lambda) + \hat{e}(-s_1, -s_2|\lambda)\Big) =$$

$$= e^{\frac{(1+\lambda)(s_1^2+s_2^2)}{2(1-\lambda)}} \oint_{u=0} du \frac{\lambda s_2 \big(u^2(1-\lambda)^2 + \lambda s_2^2\big)}{(\lambda^2 s_2^2 - u^2(1-\lambda)^2)(s_2^2 - u^2(1-\lambda)^2)} \sinh\left(\frac{\lambda s_1 s_2^2}{u(1-\lambda)^2} + s_1 u\right)$$

Rescaling then $u \mapsto u/s_1$ and denoting $S = s_1 s_2/(1-\lambda)$, one finds

$$\hat{e}_{odd}(s_1, s_2|\lambda) = \frac{\lambda}{1-\lambda} e^{\frac{(1+\lambda)(s_1^2+s_2^2)}{2(1-\lambda)}} \oint_{u=0} du \frac{S\big(u^2 + \lambda S^2\big)}{(\lambda^2 S^2 - u^2)(S^2 - u^2)} \sinh\left(\frac{\lambda S^2}{u} + u\right)$$

Using the simple identity

$$\frac{S\big(u^2 + \lambda S^2\big)}{(\lambda^2 S^2 - u^2)(S^2 - u^2)} = \sum_{k=0}^{\infty} \frac{S^{2k+1}}{u^{2k+2}} \frac{\lambda^{2k+1} - 1}{\lambda - 1}$$

we obtain a series

$$\hat{e}_{odd}(s_1, s_2|\lambda) = \frac{\lambda}{1-\lambda} e^{\frac{(1+\lambda)(s_1^2+s_2^2)}{2(1-\lambda)}} \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{\lambda^{2k+1}-1}{\lambda-1} \frac{S^{2k+2m+1}\lambda^m}{m!(m+2k+1)!}$$

and this series can be used directly to calculate the Harer-Zagier function:

$$\hat{\phi}_{odd}(s_1, s_2) = \int_0^\infty dt_1 \int_0^\infty dt_2 e^{-t_1^2/2} e^{-t_2^2/2} \hat{e}_{odd}(s_1 t_1, s_2 t_2|\lambda) =$$

$$= \frac{\lambda}{1-\lambda} \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{\lambda^{2k+1}-1}{\lambda-1} \lambda^m 4^{k+m} \left(\frac{s_1 s_2}{1-\lambda}\right)^{2k+2m+1} \frac{(m+k)!(m+k)!}{m!(m+2k+1)!} \left(\frac{1}{1-\frac{1+\lambda}{1-\lambda}s_1^2} \cdot \frac{1}{1-\frac{1+\lambda}{1-\lambda}s_2^2}\right)^{k+m+1} =$$

$$\boxed{= \frac{\lambda}{(1-\lambda)^2} \frac{\arctan \frac{s_1 s_2}{\sqrt{1-\frac{1+\lambda}{1-\lambda}(s_1^2+s_2^2)}}}{\sqrt{1-\frac{1+\lambda}{1-\lambda}(s_1^2+s_2^2)}}} \tag{52}$$

In fact, the equality between the above double series and the 2-point arctangent function is not quite obvious and requires additional comments. A more natural series representation for the 2-point arctangent function is

$$\frac{\arctan \frac{\gamma c}{\sqrt{ab-c^2}}}{\sqrt{ab-c^2}} = \sum_k \frac{(\gamma c)^{2k+1}}{(2k+1)(ab-c^2)^{k+1}} = \sum_{k,m} \frac{c^{2k2+2m+1}}{(ab)^{k+m+1}} \frac{(m+k)!}{m!k!} \frac{\gamma^{2k+1}}{2k+1} \tag{53}$$

with compact notations $\gamma = (1-\lambda)/(1+\lambda)$, $a = 1-\gamma s_1^2, b = 1-\gamma s_2^2, c = s_1 s_2$. Comparing this natural double series with the above double series expansion of the contour integral, one observes a puzzling series identity

$$4^p p! \sum_{k=0}^{p} \frac{(1-\lambda^{2k+1})\lambda^{p-k}}{(p-k)!(p+k+1)!} = \sum_{k=0}^{p} \frac{(-1)^k (1-\lambda)^{2k+1}(1+\lambda)^{2p-2k}}{k!(p-k)!(2k+1)} = \frac{1}{p!}\int_0^1 ((1+\lambda)^2 - t^2(1-\lambda)^2)^p dt \tag{54}$$

At $\lambda = 0$, this identity becomes just a B-function evaluation

$$\frac{4^p p!}{(2p+1)!} = \sum_{k=0}^{p} \frac{(-1)^k}{k!(p-k)!(2k+1)} = \frac{1}{p!}\int_0^1 (1-t^2)^p dt \tag{55}$$

while for generic $\lambda \neq 0$ it can be considered as a hypergeometric identity and proven, say, by the systematic methods of Wilf and Zeilberger [14].

The even part of the contour integral can be treated similarly. As one can see, a quite complicated contour integral gives rise to a seemingly no-less-complicated series expansion, which – after the Harer-Zagier transform! – turns into an elementary function. However, we still do not posess any general expression for these elementary functions; even derivation of the simplest examples, as it is clear from the above calculation, is not quite easy. For higher-point functions, such calculations become increasingly more complicated, and remain a challenge.

Appendix: Okounkov's formula for cubic Kontsevich model [12]

Another important direction of generalization is from the Gaussian Kontsevich model to Generalized Kontsevich models of higher degree. In the case of cubic Kontsevich model

$$Z_K = \frac{\int DX \ \exp\left(-\frac{1}{3}\mathrm{Tr}X^3 - \frac{1}{\sqrt{3}}\mathrm{Tr}AX^2\right)}{\int DX \ \exp\left(-\frac{1}{\sqrt{3}}\mathrm{Tr}AX^2\right)}, \qquad \tau_{2k+1} \equiv \frac{3^{2k+1}}{2k+1}\mathrm{Tr}A^{-2k-1} \tag{56}$$

where the conventional resolvents have a form

$$\rho_K(z_1,\ldots,z_m) = \hat{\Delta}(z_1)\ldots\hat{\Delta}(z_m)\log Z_K\Big|_{\tau=0}, \qquad \hat{\Delta}(z) = \sum_{n=0}^{\infty}\frac{1}{z^{n+3/2}}\frac{\partial}{\partial\tau_{2n+1}} \tag{57}$$

a contour integral representation was obtained by Okounkov (see [12] for more details) for the exponential functions $e_K(x_1,\ldots,x_k)$, which are connected with the above resolvents by Laplace transform:

$$\rho_K(z_1,\ldots,z_k) = 2^k \int\limits_0^\infty \prod_{i=1}^{k} ds_i e^{-\sum\limits_i s_i z_i} e_K(s_1,\ldots,s_k) \tag{58}$$

This contour integral representation has a form

$$\begin{aligned} e_K(s) &= \mathcal{E}(s), \\ e_K(s_1,s_2) &= \mathcal{E}(s_1,s_2) - \mathcal{E}(s_1)\mathcal{E}(s_2), \\ e_K(s_1,s_2,s_3) &= \mathcal{E}(s_1,s_2,s_3) - \mathcal{E}(s_1)\mathcal{E}(s_2,s_3) - \mathcal{E}(s_2)\mathcal{E}(s_1,s_3) - \mathcal{E}(s_3)\mathcal{E}(s_1,2) + \mathcal{E}(s_1)\mathcal{E}(s_2)\mathcal{E}(s_3), \\ &\ldots \end{aligned} \tag{59}$$

and so on, where

$$\mathcal{E}(s_1,\ldots,s_n) = \frac{1}{2^n\pi^{n/2}}\prod_{\alpha=1}^{n}\frac{e^{\frac{1}{12}s_\alpha^3}}{\sqrt{s_\alpha}}\int_{t_\alpha\geq 0} dt_\alpha \exp\left(-\frac{(t_\alpha - t_{\alpha+1})^2}{4s_\alpha} - \frac{(t_\alpha + t_{\alpha+1})s_\alpha}{2}\right) \tag{60}$$

These formulas can be considered as direct generalisations of Brezin-Hikami contour integrals for the Gaussian Kontsevich model. It is puzzling to ask, whether similar integral representations exist for arbitrary monomial potentials in GKM, or maybe even for generic non-monomial potentials. Of interest is also to find analogues of elementary multidensities in these models.

Acknowledgements

Our work is partly supported by Russian Federal Nuclear Energy Agency, Federal Agency for Science and Innovations of Russian Federation under contract 02.740.11.5194, by RFBR grants 10-01-00536, by joint grants 09-02-90493-Ukr, 09-02-93105-CNRSL, 09-01-92440-CE, 09-02-91005-ANF, 10-02-92109-Yaf-a, by CNRS (A.M.) and by Dynasty Foundation (Sh.Sh.).

References

[1] A.Morozov, Phys.Usp.(UFN) **35** (1992) 671-714; **37** (1994) 1, hep-th/9303139; hep-th/9502091; hep-th/0502010
A.Mironov, Int.J.Mod.Phys. **A9** (1994) 4355, hep-th/9312212; Phys.Part.Nucl. **33** (2002) 537; hep-th/9409190; Theor.Math.Phys. **114** (1998) 127, q-alg/9711006

[2] A.Alexandrov, A.Mironov and A.Morozov, Int.J.Mod.Phys. **A19** (2004) 4127, hep-th/0310113

[3] J. Harer, D. Zagier, Inv. Math. **85** (1986) 457-485

[4] Sh.Shakirov and A.Morozov, JHEP 0912(2009)003, arXiv:0906.0036

[5] M. Kontsevich, Commun. Math. Phys. **147** (1992) 1; Funk. Anal. i Pril. **25** (1991) 50;
S. Kharchev, A. Marshakov, A. Mironov, A. Morozov and A. Zabrodin, Phys. Lett. B **275** (1992) 311, hep-th/9111037; Nucl. Phys. B **380** (1992) 181, hep-th/9201013;
C. Itzykson and J. B. Zuber, Int. J. Mod. Phys. A **7** (1992) 5661, hep-th/9201001

[6] L.Chekhov and Yu.Makeenko, Phys.Lett. B278 (1992) 271-278, arXiv:hep-th/9202006
S. Kharchev and A. Marshakov, hep-th/9210072;
S. Kharchev and A. Marshakov, Int. J. Mod. Phys. A **10** (1995) 1219, hep-th/9303100
A. Mironov, A. Morozov and G. W. Semenoff, Int. J. Mod. Phys. A **11** (1996) 5031, hep-th/9404005
A.Alexandrov, A.Mironov and A.Morozov, JHEP 0912(2009)053, arXiv:0906.3305

[7] E.Brezin and S.Hikami, JHEP 0710(2007)096, arXiv:0709.3378; Commun.Math.Phys.283(2008)507-521, arXiv:0708.2210; arXiv:cond-mat/9804024

[8] N.Nekrasov, Adv.Theor.Math.Phys. **7** (2004) 831-864
G.Moore, N.Nekrasov, S.Shatashvili, Nucl.Phys. **B534** (1998) 549-611, hep-th/9711108; hep-th/9801061
A.Losev, N.Nekrasov and S.Shatashvili, Commun.Math.Phys. **209** (2000) 97-121, hep-th/9712241; ibid. 77-95, hep-th/9803265

[9] D.Gaiotto, arXiv:0908.0307
L.Alday, D.Gaiotto and Y.Tachikawa, Lett.Math.Phys. **91** (2010) 167-197, arXiv:0906.3219
N.Wyllard, JHEP **0911** (2009) 002, arXiv:0907.2189
D.Nanopoulos and D.Xie, arXiv:0908.4409; JHEP **1003**(2010)043, arXiv:0911.1990
A.Marshakov, A.Mironov and A. Morozov, arXiv:0907.3946; Phys.Lett.**B682** (2009) 125-129, arXiv:0909.2052; JHEP **0911** (2009) 048, arXiv:0909.3338
A. Mironov and A. Morozov, Nucl.Phys.**B825**(2010)1-37, arXiv:0908.2569; Phys.Lett.**B682**(2009)118-124, arXiv:0909.3531
N.Nekrasov and S.Shatashvili, Nucl.Phys.B, Proc.Suppl. **192-193** (2009) 91-112; arXiv:0901.4748, arXiv:0908.4052
N.Nekrasov and E.Witten

[10] A.Mironov, A.Morozov and And.Morozov, arXiv:1003.5752

[11] Vl.Dotsenko and V.Fateev, Nucl.Phys. **B240** (1984) 312-348
A.Gerasimov, A.Marshakov, A.Morozov, M.Olshanetsky, S. Shatashvili, Int.J.Mod.Phys. **A5** (1990) 2495
A.Gerasimov, A.Marshakov and A.Morozov, Nucl.Phys. **B328** (1989) 664, Theor.Math.Phys. **83** (1990) 466-473; Phys.Lett. **B236** (1990) 269, Sov.J.Nucl.Phys. **51** (1990) 371-372
A. Marshakov, A. Mironov, and A. Morozov, Phys.Lett.B265(1991)99-107
S. Kharchev, A. Marshakov, A. Mironov, A. Morozov and S. Pakuliak, Nucl.Phys.B404 (1993) 717-750, hep-th/9208044
A.Mironov and A. Morozov, Phys.Lett.**B680**(2009)188-194, arXiv:0908.2190

R. Dijkgraaf and C. Vafa, arXiv:0909.2453
H.Itoyama, K.Maruyoshi and T.Oota, arXiv:0911.4244
T. Eguchi and K. Maruyoshi, arXiv:0911.4797
R. Schiappa and N. Wyllard, arXiv:0911.5337
A.Mironov, A.Morozov and Sh.Shakirov, JHEP **1002** (2010) 030, arXiv:0911.5721; arXiv:1001.0563; arXiv:1004.2917

[12] A.Okounkov, arXiv:math/0101201
A.Alexandrov, A.Mironov, A.Morozov and P.Putrov, Int.J.Mod.Phys.**A24**(2009)4939-4998, arXiv:0811.2825

[13] N.Nekrasov, Adv.Theor.Math.Phys.**7** (2004) 831-864, hep-th/0206161
R.Flume and R.Poghossian, Int.J.Mod.Phys. **A18** (2003) 2541, hep-th/0208176
S. Shadchin, arXiv:hep-th/0601167

[14] M. Petkovsek, H. Wilf and D. Zeilberger, *"$A = B$"*, 1996.